The Inverse-Orthogonal Protocol: A Unified Algebraic Strategy for Geometric Locus Problems

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ABSTRACT
Students in upper-secondary education often experience high cognitive load when solving geometric transformation problems, particularly the rotation of curves (loci). Recent studies indicate a substantial pedagogical discontinuity between the rotation of points (forward mapping) and the rotation of equations (inverse mapping). This paper proposes a unified algorithmic approach leveraging the orthogonality property of rotation matrices ($R^{-1} = R^T$). By substituting the inverted coordinates directly into the initial locus equation, we demonstrate a robust method that handles linear equations, parabolas, and general conics with equal ease. This protocol effectively manages non-standard angles and automatically accounts for the xy interaction term in rotated conics, eliminating the need for rote memorization of transformation formulas.

1 INTRODUCTION

Geometric transformation is a cornerstone of the upper-secondary school mathematics curriculum. Although students generally demonstrate proficiency in rotating individual points $(x,y) \to (x',y')$, a systemic failure often occurs when the task transitions to finding the equation of a rotated curve.

Recent educational research by Hernandez-Zavaleta et al. (2026) highlights that traditional teaching methods often fail to bridge the gap between "visual" geometry and "abstract" algebra. The root of this difficulty lies in the directionality of the function:

- **Forward Process:** Rotating a point uses $\mathbf{v}' = R\mathbf{v}$.
- **Inverse Process:** Finding a new equation requires $\mathbf{v} = R^{-1}\mathbf{v}'$.

Conventional textbooks often treat these as separate skills, leading to significant conceptual gaps and substitution errors during high-stakes examinations [2]. Students frequently confuse when to use $+\alpha$ or $-\alpha$.

This paper introduces the **Inverse-Substitution Method**. By leveraging the linear algebra concept of orthogonality, we provide a single, consistent algorithm that works for any curve type.

2 THEORETICAL FRAMEWORK

2.1 The Rotation Matrix as an Orthogonal Operator

A rotation of the Euclidean plane by an angle α is represented by the matrix R_α:

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1}$$

A fundamental property often omitted in secondary education is that R_α is an *orthogonal matrix*. This implies that its inverse is equivalent to its transpose [3]:

$$R^{-1} = R^T = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{2}$$

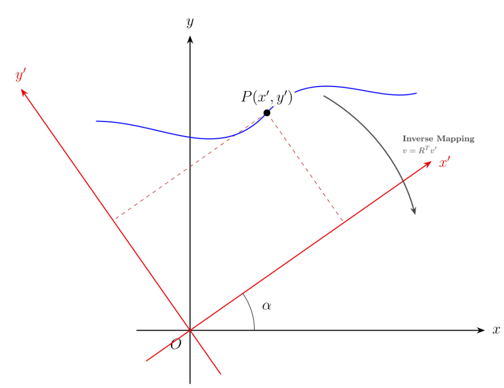

Figure 1: Visualizing the Inverse Mapping. To find the locus of the new curve, we map any point (x',y') back to the original coordinate system using R^T.

2.2 Derivation of Substitution Formulas

To derive the substitution equations, we begin with the standard rotation relation $\mathbf{v}' = R_\alpha\mathbf{v}$. To isolate the original coordinates $\mathbf{v}$, we multiply both sides by R^{-1} (which is R^T):

$$\mathbf{v} = R_\alpha^T \mathbf{v}'$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{3}$$

Performing the matrix multiplication yields the explicit system:

$$x = x'\cos\alpha + y'\sin\alpha \tag{4}$$
$$y = -x'\sin\alpha + y'\cos\alpha \tag{5}$$

These equations are the core of our protocol. Note that the sign of the sine term effectively reverses, algebraically encoding the "inverse" rotation.

3 METHODOLOGY

We propose a streamlined three-step protocol for classroom implementation.

Step 1: Define Inverse Map
Identify $\sin\alpha$ and $\cos\alpha$. Construct the relations based on R^T:

$$x = x'\cos\alpha + y'\sin\alpha, \quad y = -x'\sin\alpha + y'\cos\alpha$$

Step 2: Substitution
Substitute these expressions directly into the variable slots of the initial locus equation $f(x,y) = 0$.

Step 3: Simplification
Expand and simplify the algebraic equation to obtain the final form in terms of x' and y'.

4 CASE STUDIES

4.1 Case A: The Pythagorean Rotation

Problem: Rotate the line $y = 2x+1$ by an angle α where $\tan\alpha = 4/3$ (a classic 3-4-5 triangle).
Solution: Here, $\sin\alpha = 4/5$ and $\cos\alpha = 3/5$. Substituting into $y = 2x+1$:

$$\left(-\frac{4}{5}x' + \frac{3}{5}y'\right) = 2\left(\frac{3}{5}x' + \frac{4}{5}y'\right) + 1 \tag{6}$$

Multiplying by 5 to clear denominators:

$$-4x' + 3y' = 6x' + 8y' + 5 \tag{7}$$

Rearranging terms yields $10x' + 5y' + 5 = 0$, which simplifies to:

$$2x' + y' + 1 = 0 \tag{8}$$

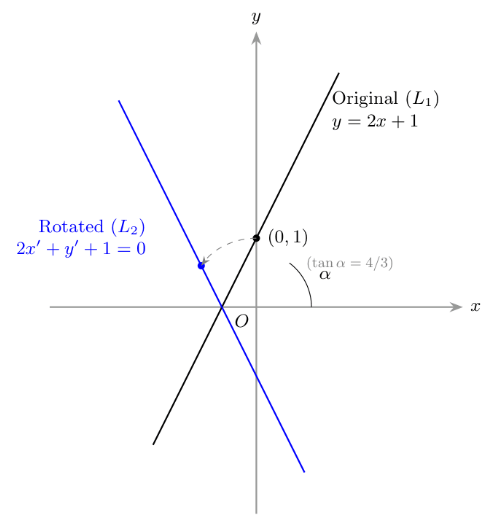

Figure 2: Rotation of the line $y = 2x+1$. The Inverse-Orthogonal protocol handles Pythagorean triple inputs seamlessly.

4.2 Case B: The Rotated Parabola

Problem: Rotate the standard parabola $y = x^2$ by 45°.
Solution: For 45°, $\sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}}$.

$$x = \frac{x'+y'}{\sqrt{2}}, \quad y = \frac{-x'+y'}{\sqrt{2}} \tag{9}$$

Substituting into $y = x^2$:

$$\frac{-x'+y'}{\sqrt{2}} = \left(\frac{x'+y'}{\sqrt{2}}\right)^2 \tag{10}$$

Expanding the square term and multiplying by 2:

$$\sqrt{2}(-x'+y') = x'^2 + 2x'y' + y'^2 \tag{11}$$

This derivation effortlessly produces the $x'y'$ interaction term, which is often a source of confusion in standard pedagogy [4].

5 CONCLUSION

The Inverse-Substitution Method offers a superior pedagogical strategy for locus problems. It unifies the approach for lines and conics, eliminates confusion regarding rotation direction, and provides a robust framework for handling complex algebraic structures. We recommend this method be adopted as the standard protocol in advanced mathematics classrooms.

References

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